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MINIMAL SURFACES APPLICABLE TO SURFACES OF REVOLUTION.

By J. K. Whittemore.

§ 1. **Introduction.** One of the simplest and most interesting classes of minimal surfaces is composed of those which are applicable to surfaces of revolution. The treatises on differential geometry give the determination of these surfaces and some of their more obvious properties. There have appeared since their first determination in 1862 several papers in mathematical journals dealing with these surfaces. It is the purpose of this article in Part I to give an account of these papers,* together with a new theorem regarding conjugate systems, and in Part II to prove certain theorems concerning them, believed to be new, relating to their symmetries and to their double lines, and finally to state without proof a new property characteristic of these surfaces.

The Enneper-Weierstrass equations of a real minimal surface are

$$x = \frac{1}{2}\int (1 - u^2)F(u)du + \frac{1}{2}\int (1 - v^2)\phi(v)dv,$$

$$y = \frac{i}{2}\int (1 + u^2)F(u)du - \frac{i}{2}\int (1 + v^2)\phi(v)dv,$$

$$z = \int uF(u)du + \int v\phi(v)dv,$$

where u, v are conjugate variables, and $F(u)$, $\phi(v)$ are conjugate analytic functions. Minimal surfaces applicable to surfaces of revolution are given by

$$F(u) = Cu^{m-2},$$

where m is real. We shall denote these surfaces, following Haag, as B surfaces, or B_m if it is desired to emphasize the value of m; this notation is suggested by the fact that the surfaces were first determined by E. Bour.

If $m \neq 0$ we may without restriction choose $C = 1$ in $F(u)$. It may be shown that m and $-m$ give the same surface, so that we may suppose $m \geqq 0$. Special surfaces of particular interest are $m = 0$, $C = 1$, the catenoid; $m = 0$, $C = i$, the right helicoid; $m = 2$, Enneper's surface.

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* Exact references to the treatises and papers considered are given in § 2 of this article.

Excluding the cases $m = 0$ and $m = 1$, the equations of B_m are

$$x = \frac{1}{2}\left(\frac{u^{m-1}}{m-1} - \frac{u^{m+1}}{m+1}\right) + \frac{1}{2}\left(\frac{v^{m-1}}{m-1} - \frac{v^{m+1}}{m+1}\right),$$

$$y = \frac{i}{2}\left(\frac{u^{m-1}}{m-1} + \frac{u^{m+1}}{m+1}\right) - \frac{i}{2}\left(\frac{v^{m-1}}{m-1} + \frac{v^{m+1}}{m+1}\right),$$

$$z = \frac{u^m + v^m}{m}.$$

The direction cosines of the normal are

$$X = \frac{u+v}{uv+1}, \qquad Y = \frac{i(v-u)}{uv+1}, \qquad Z = \frac{uv-1}{uv+1}.$$

The first and second fundamental forms are respectively

$$ds^2 = (uv+1)^2(uv)^{m-2}dudv, \qquad \Sigma Xd^2x = -\,u^{m-2}du^2 - v^{m-2}dv^2.$$

The differential equation of the lines of curvature and of the asymptotic lines are respectively

$$u^{m-2}du^2 - v^{m-2}dv^2 = 0, \qquad u^{m-2}du^2 + v^{m-2}dv^2 = 0.$$

The total curvature is

$$K = \frac{-4}{(uv+1)^4(uv)^{m-2}}.$$

We shall frequently use the real variables, r and φ, where $u = re^{\phi i}$ and $v = re^{-\phi i}$. With these variables the equations are

$$x = \frac{r^{m-1}}{m-1}\cos(m-1)\varphi - \frac{r^{m+1}}{m+1}\cos(m+1)\varphi,$$

$$y = -\frac{r^{m-1}}{m-1}\sin(m-1)\varphi - \frac{r^{m+1}}{m+1}\sin(m+1)\varphi,$$

$$z = \frac{2r^m}{m}\cos m\varphi,$$

$$X = \frac{2r\cos\varphi}{r^2+1}, \qquad Y = \frac{2r\sin\varphi}{r^2+1}, \qquad Z = \frac{r^2-1}{r^2+1}, \qquad K = \frac{-4}{(r^2+1)^4r^{2m-4}}.$$

Both sets of expressions for X, Y, Z hold for every minimal surface, and the curves (r), that is $r =$ constant, are the Minding parallels of the surface, curves along which the plane tangent to the surface is everywhere equally inclined to the xy plane; the curves (φ) are the Minding meridians, curves along which the normal to the surface is parallel to a fixed plane containing the Z axis. Much of the interest of the surfaces B is centered on the curves (r) and (φ). For any minimal surface their spherical representations are the meridians and parallels of the sphere; since for B K is constant along (r), these curves correspond to the parallels of the applicable surface of revolution; the curves (φ) are the orthogonal trajectories of (r), therefore correspond to the meridians of the surface of revolution, and are in consequence geodesic lines of B.

I.

§ 2. **Literature. Papers of Bour and Schwarz. Theorem on Conjugate Systems.** A discussion of the surfaces B is given by Darboux, Théorie des Surfaces, vol. I (2d ed.), pp. 367, 368, 392–395, with particular consideration of the " minimal helicoids," $m = 0$, p. 359, and of Enneper's surface, pp. 372–376; the Minding parallels and meridians are explained on p. 357. Shorter treatments are given by Eisenhart, Differential Geometry, pp. 328–330, 260, 267, 269, and by Bianchi, Lezioni di Geometria Differenziale, vol. II (2d ed.), pp. 321–324.

The following papers deal wholly or in part with surfaces B:

E. Bour, " Théorie de la déformation des surfaces." Journal de l'École Polytechnique, cahier 39 (1862), pp. 99–109.

H. A. Schwarz, " Miscellen aus dem Gebiete der Minimalflächen." Journal de Crelle, vol. 80 (1875), p. 295, published also in Gesammelte Mathematische Abhandlungen.

A. Ribaucour, " Étude sur les élassoides ou surfaces à courbure moyenne nulle." Mémoires Couronnés de l'Académie Royale de Belgique, vol. XLIV (1882), chapter XX, pp. 215–224.

A Demoulin, Bulletin des Sciences Mathematiques (2), vol. XXI (1897), pp. 244–252.

J. Haag, same Bulletin (2), vol. XXX (1906), pp. 75–94, also pp. 293–296.

E. Stübler, Mathematische Annalen, vol. 75 (1914), pp. 148–176.

Only small parts of the first three papers cited deal with the surfaces B. Demoulin's paper is chiefly and Haag's two papers are wholly devoted to them; Stübler's paper is occupied chiefly with related surfaces. We consider these papers in the order given.

Bour determines first the surfaces of revolution applicable to minimal surfaces. These surfaces of revolution are of little interest. He then determines the minimal surfaces corresponding. As previously stated, this is the first determination. He gives their equations with the coördinates r and φ. The Enneper-Weierstrass equations resulted from work of Enneper published in 1864 and of Weierstrass published in 1866, so were not available to Bour. The only properties of these minimal surfaces given by Bour are contained in the following three theorems, stated by him without proof: (1) Every geodesic (φ) cuts at a constant angle all curves of each family of the lines of curvature, of each family of asymptotic lines, and of the family of level curves (z); (2) at every point the lines of curvature bisect the angles of the curves (z) and (r), and the asymptotic lines bisect the angles of the curves (z) and (φ); (3) the developable tangent to B along every curve (φ) is a cylinder whose elements are parallel to the xy plane. All of these theorems may be easily verified, but only the first is of interest for, as appears from a general theorem which we state and prove in the next paragraph, the second theorem is true for all minimal surfaces while the third theorem is true for all non-developable surfaces if in the statement "Minding meridian" be substituted for "curve (φ)."

The theorem, which we believe to be new, is as follows: On every surface, not developable, the level curves (z) and the Minding meridians (Y/X) form a conjugate system; the orthogonal trajectories of the level curves, sometimes called the curves of steepest ascent, and the Minding parallels (Z) also form a conjugate system.

Both parts of this theorem may be easily proved geometrically; the first part indeed is a limiting case of Koenig's well-known theorem, which is stated by Darboux* as follows: The sections of a surface made by the planes through a straight line D form a conjugate system with the curves of contact with the surface of cones whose vertices lie on D. If we suppose D to lie in the xy plane and to recede to infinity without changing its direction the first part of our theorem is obtained. The two parts may easily be proved analytically. The proof follows.

The condition for conjugate directions is, with the usual notation, $\Sigma dx\delta X = 0$, where d and δ are differentials in the two directions. For a level curve $dz = 0$, and for a Minding meridian $\delta(Y/X) = 0$, from which we have

$$\frac{\delta X}{X} = \frac{\delta Y}{Y}.$$

From the equation $\Sigma X dx = 0$ we have, if the surface is not developable, substituting $dz = 0$ and replacing X and Y by δX and δY, $\Sigma dx\delta X = 0$. To prove the second part of the theorem we denote by d_1 and δ_1 the differentials along the curve of steepest ascent and the Minding parallel respectively, so that

$$\Sigma dx d_1 x = dx d_1 x + dy d_1 y = 0, \qquad \delta_1 Z = 0.$$

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* Vol. I (2d ed.), p. 164.

We have to prove

$$\Sigma d_1x\delta_1X = d_1x\delta_1X + d_1y\delta_1Y = 0.$$

The last equation is easily verified if we note that

$$dz = pdx + qdy = 0, \qquad \delta_1X\delta_1q - \delta_1Y\delta_1p = 0,$$

the second of these holding good since

$$\delta_1Z = \frac{p\delta_1p + q\delta_1q}{(1 + p^2 + q^2)^{3/2}} = 0.$$

From the theorem just proved we may easily get the result obtained by Minding* in the paper in which are first considered the curves that we have called the Minding parallels and meridians. Minding proved that the only surfaces, not developable, on which these parallels and meridians form an orthogonal system are the molding surfaces, including surfaces of revolution, whose axis is the Z axis, and the minimal surfaces. We obtain these results at once: If the Minding parallels and meridians form an orthogonal system suppose first that at every point one of the directions d, d_1, δ, δ_1 coincides with that of a line of curvature at the point; then the four directions coincide with the directions of the two lines of curvature and the level curves (z) form a family of lines of curvature, and the surface is a molding surface whose axis is the Z axis.† If none of the four directions coincides with that of a line of curvature we may call the slopes of the tangents to the curves of differentials d, d_1, δ, δ_1 referred to the principal axes of the indicatrix at the point m, m_1, μ, μ_1 respectively, and have

$$mm_1 = -1, \qquad m\mu = m_1\mu_1 = e^2 - 1,$$

where e is the eccentricity of the indicatrix. Then the necessary and sufficient condition that the directions of δ and δ_1 be orthogonal is

$$\mu\mu_1 = -(e^2 - 1)^2 = -1$$

and $e^2 = 0$ or 2, so that the indicatrix is at every point a circle or a rectangular hyperbola, and the surface is a sphere or a minimal surface.

Returning to the consideration of the papers mentioned at the beginning of this section we find in Schwarz's Miscellen the proof that all real minimal surfaces applicable to surfaces of revolution are given by setting $F(u) = Cu^{m-2}$, where m is real, in the Enneper-Weierstrass equations. His rather brief proof is elaborated by Darboux.‡ Schwarz was the first to prove that any real minimal surface applicable to the minimal surface given by the Enneper-Weierstrass equations is obtained by substituting for $F(u)$ in these equations $e^{i\alpha}F(u)$, where α is a real constant.* The surfaces so obtained are now known as the *associate surfaces* of the given minimal surface. The value $\alpha = \pi/2$ gives the *adjoint surface.* G. Scheffers states† that K. Peterson had observed earlier that all associate minimal surfaces are applicable.

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* Journal de Crelle, vol. 44 (1852), p. 66.

† See Eisenhart, p. 308.

‡ Vol. I (2d ed.), pp. 392–395.

§ 3. **Papers of Ribaucour, Demoulin, Haag, and Stübler.** In Ribaucour's memoir a chapter is devoted to the surfaces B. Ribaucour states that this investigation was made at his request by Rouquet. We find here the first full study of the geometry of these surfaces. Throughout this work the values $m = 0$ or 1 must be excluded. The coördinates r, φ are used almost exclusively. No use is made of the Enneper-Weierstrass equations and Schwarz's determination $F(u) = Cu^{m-2}$. There appear all of Bour's results, some of them in different form. The new results of chief interest follow.

A surface B is the envelope of a family of cylinders, *similar* to each other, whose elements are parallel to the xy plane, and whose characteristics are the curves (φ).

A surface B_m is composed of congruent nappes, each containing a plane of symmetry—these planes are actually those of certain plane geodesics (φ) (see Darboux, loc. cit., p. 460)—obtained from one by rotation about the Z axis through successive angles $2\pi/m$. If $m = p/q$, where p and q are integers having no common factor, the number of congruent nappes is p. This number is m if m is an integer. If m is rational, B_m is algebraic. The length of an arc of a geodesic (φ) is determined. All surfaces associate to B are congruent to B. Ribaucour calls these associate surfaces " élassoides groupés," and proves many of their properties, but makes no reference to Peterson or Schwarz. It is not however proved here that all associate surfaces are obtained by rotating B about the Z axis. This fact was proved by Darboux, loc. cit., p. 395. Ribaucour falls into an error‡ in stating and attempting to prove that two applicable surfaces having parallel tangent planes at corresponding points must be *adjoint* minimal surfaces (" élassoides conjugués "). It is well known that any two *associate* minimal surfaces have this property, and Darboux has proved that there are no others—loc. cit., p. 383.

Ribaucour shows that if $m = p/q$ the class of B_m is $2q(p + q)$; if m is integral the degree is $(m + 1)^2$. Each curve (r) lies on a quadric of revolution:

$$x^2 + y^2 + \frac{m^2}{m^2 - 1} z^2 = \left(\frac{r^{m-1}}{m - 1} + \frac{r^{m+1}}{m + 1}\right)^2 .$$

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* "Bestimmung einer speciellen Minimalfläche," Ges. Math. Abh., vol. I, p. 119.

† Theorie der Flächen, vol. II (2d ed.), p. 360.

‡ P. 106.

Each geodesic (φ) lies on a cone of the second degree. It is stated and incorrectly proved that if m is integral the degree of the curves (φ) is $2m - 1$. The correct value is, as given by Haag, $m + 1$. Ribaucour's error may perhaps be understood if it is noticed that $2m - 1$ gives the correct value for the simplest case, $m = 2$, that is Enneper's surface. It is of interest to examine Ribaucour's incorrect proof, as it is an example of an error into which one may easily fall; the proof even when corrected is unsatisfactory. It is as follows:

Eliminating r from the equations of B_m we have, changing to the notation of this paper,

$$[y \cos (m - 1)\varphi + x \sin (m - 1)\varphi][y \cos (m + 1)\varphi - x \sin (m + 1)\varphi] = \frac{m^2 z^2 \sin^2 m\varphi}{m^2 - 1},$$

$$2[y \cos (m + 1)\varphi - x \sin (m + 1)\varphi]^m = \frac{(mz \sin m\varphi)^{m-1} \sin 2m\varphi}{(m - 1)^m}.$$

The curve (φ) lies on each of the surfaces given by these two equations, where φ is regarded as constant, of degree 2 and m respectively, and is therefore their intersection, or a part of their intersection. Ribaucour notes that these surfaces have in common the straight line,

$$y \cos (m + 1)\varphi - x \sin (m + 1)\varphi = z = 0,$$

and asserts in consequence that the curve (φ) is of degree $2m - 1$. The straight line however is a multiple line of the second surface of multiplicity $m - 1$, and must be counted as $m - 1$ lines in the intersection of the two surfaces, so that the degree of the remaining part of the intersection is $2m - (m - 1) = m + 1$, which is indeed the correct value of the degree of (φ). But though this result is correct the proof is unsatisfactory, for we cannot be sure that the two surfaces have not other intersections in addition to the straight line and the curve (φ). The simplest and the only reliable method of determining the degree of (φ) is to regard the equations of the surface as parametric equations of (φ) with parameter r, from which it appears immediately that such a curve intersects the general plane in $m + 1$ points when m is integral.

Among the geodesics (φ) are certain plane curves, all congruent. Enneper's surface is of degree 9 and class 6; $B_{1/2}$ is of degree 12 and class 12.

The paper of Demoulin is divided into three parts. In the first is given a simple proof of a new property characteristic of surfaces B: The necessary and sufficient condition that a minimal surface be the middle envelope of a rectilinear congruence and that the developables of the congruence correspond to the lines of curvature of the surface is that the latter be applicable to a surface of revolution. In this case the congruence is a " Congruence of Ribaucour " and the director surface is the adjoint minimal surface.* Demoulin's proof is based on an interesting theorem proved very easily by him with the aid of the moving trihedral:† In order that a surface be applicable to a surface of revolution it is necessary and sufficient that there exist in the tangent plane at each point M of the surface a point O such that the infinitesimal displacement of O corresponding to any infinitesimal displacement of M be parallel to the plane determined by the normal at M and the displacement of M.

The second part of the paper gives a determination of the functions $F(u)$ and $\phi(v)$ by means of identifying the linear element with the known form for a surface of revolution, $ds^2 = f(u - v)dudv$. The most interesting thing in Demoulin's paper is the finding of certain *imaginary* minimal surfaces applicable to surfaces of revolution, not previously known. These are given by the Enneper-Weierstrass equations where

$$F(u) = Ce^{mu}, \qquad \phi(v) = C_1 \frac{e^{m/v}}{v^4}.$$

That these imaginary surfaces were not discovered by the method of Schwarz and Darboux may be explained by remarking that their method is based on geometrical intuition which can be applied only to real surfaces.

In the last part of this paper the determination of minimal surfaces applicable to a spiral surface is made to depend in a simple way on the preceding work.

The two papers of Haag contain new and interesting results, and give evidence of much labor, but are open to some criticism.

The first paper is devoted to a new determination of the surfaces B, by means of the moving trihedral, and to a study of the properties of these surfaces by the same method and by the ordinary methods of differential geometry. We cannot avoid the feeling that much of this paper is rather an exercise in the employment of the method of the moving trihedral than a useful application of the method. The author proves various properties of the B surfaces previously given, as he states, by Bour, but seems not to be acquainted with the work of Ribaucour for he gives, apparently as new, several theorems contained in the latter's memoir. Among Haag's results are the following: The surfaces B_m and B_{-m} are the same; among the geodesics (φ) are certain straight lines; all geodesics (φ) and no other geodesics are helices; the helicoidal developables tangent to B along curves (r) cut the xy plane in hypocycloids, homothetic with respect to the origin; consequently the edge of regression of each of these developables is an evolute of a hypocycloid; the cusps of these hypocycloids lie on the straight lines (φ), their vertices on the traces on the xy plane of the planes of the plane geodesic lines of curvature (φ).

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* See Eisenhart, pp. 420–422.

† Bulletin de la Société mathematique de France, vol. XXII (1894), p. 47.

Concerning the helicoidal developable we remark that it is the envelope of a one-parameter family of planes equally inclined to a fixed plane; its edge of regression is a helix whose axis is perpendicular to the fixed plane; the orthogonal trajectories of the straight lines of the surface, that is the involutes of the edge of regression, are the plane sections of the surface parallel to the fixed plane; conversely the surface of tangents to any helix is a helicoidal developable. Evidently the developable tangent to any surface along a Minding parallel is a helicoidal developable. These are the surfaces called in German "Böschungsflächen."

For $r = 1$ this developable is a cylinder whose elements are parallel to the Z axis. Haag remarks that the apparent contour of B on the xy plane is therefore a hypocycloid, but even the simplest B surface, Enneper's surface, has no apparent contour on the xy plane as is at once evident from an examination of Brill's familiar model of that surface: all the elements of the cylinder, $r = 1$, cut the surface. We remark that for Enneper's surface the hypocycloids are four cusped. Haag does not observe that if $m < 1$ hypocycloid must in all these statements be replaced by *epicycloid*. He shows also that the cone tangent to B whose vertex is the origin is of revolution, and is real only if $m < 1$.

It is evident that the properties of B_m differ considerably in the two cases, m greater or less than one. We may here record one such property which seems not to have been previously noticed: The surface B_m admits a real geodesic (r) when and only when $m < 1$, namely,

$$r = \sqrt{\frac{1 - m}{1 + m}}.$$

Haag's second paper is devoted to the determination of the degree and class of B_m and of certain curves on this surface when the surface is algebraic, that is when $m = p/q$. Several of his results are contained in Ribaucour's paper. We find that certain of his new results are incorrect.

His determination of the class of B_m agrees with Ribaucour's. The degree of B_m is given by the formula

$$d = \mu^2 - \omega,$$

where μ is the degree of the minimal curve Γ whose translation generates the surface, and ω is the number of points at infinity in which Γ meets its conjugate. The equations of Γ may be written, if we set $u^{1/q} = \nu$,

$$x = \frac{1}{2}\left(\frac{\nu^{p-q}}{m-1} - \frac{\nu^{p+q}}{m+1}\right), \qquad y = \frac{i}{2}\left(\frac{\nu^{p-q}}{m-1} + \frac{\nu^{p+q}}{m+1}\right), \qquad z = \frac{\nu^p}{m}.$$

If $m > 1$, that is if $p > q$, the degree of Γ is $p + q$; it is easily seen that in this case $\omega = 0$, and $d = (p + q)^2$, equal, when m is an integer, to $(m + 1)^2$, agreeing with Ribaucour. But if $m < 1$, that is if $p < q$, the degree of Γ is not $p + q$, as stated, but $2q$. Further Haag's determination of ω for this case seems incorrect; for the multiplicity of the point of Γ at infinity given by $\nu = \infty$ is q, and that of the point at infinity given by $\nu = 0$ is $q - p$, so that $\omega = 2q(q - p)$, not $q^2 - p^2$. Then d is not $2p(p + q)$, as given by Haag, but

$$d = 4q^2 - 2q(q - p) = 2q(p + q).$$

We note that the latter formula gives the degree of $B_{1/2}$ as 12, as stated by Ribaucour. It is remarkable that if $m < 1$ the class and degree of B_m are aways the same. The degree is always greater than the class if $m > 1$. The degree of the curves (r) is $2(p + q)$, of the geodesics (φ) $p + q$ if $m > 1$, $2q$ if $m < 1$. Haag gives the latter as $p + q$ in both cases.

Stübler's paper is very interesting, but is so compact that it seems impossible to give more than an unsatisfactory outline of its contents. We regret that he has not noticed the papers of Haag. Only a short section is given to the surfaces B, the greater part of the paper being devoted to the discussion of related surfaces.

The first part contains various theorems previously known and some new results, chiefly relating to the helicoidal developables tangent to B along the curves (r), in particular results connected with their development on a plane and certain related cycloidal curves. It appears from his work that the projections on the xy plane of the curves (r) are hypo- or epitrochoids. He gives a new property characteristic of the surfaces B, which is easily proved: The only real minimal surfaces having a real curve along which the total curvature is a maximum are the surfaces B_m, $|m| \leqq 2$; the curve is $r = \sqrt{\frac{2-m}{2+m}}$.

The second section contains a discussion of the surfaces of center, or evolute surfaces, of B. The most interesting result is that the curve on each nappe corresponding to the curve of maximum total curvature on B is a singular curve of constant curvature.

In the third section are considered those involute surfaces of B whose normals are the tangents of the geodesics (φ). These are W surfaces and have one family of plane lines of curvature—this appears at once from the consideration that all involutes of a helix are plane. The lines of curvature of the second family are spherical. If $m = p/q$ the degree of these surfaces is $2q(p + q)$ equal to the class of B_m, and, if $m < 1$, to the degree of B_m. When $m > 1$ this degree is less than that of B_m.

In the fourth section the cases, $m = 1$ and 0, previously excluded are taken up, and their evolute and involute surfaces discussed.

In the second part of the paper Stübler determines all minimal surfaces having certain properties in common with the surfaces B. The first section of the second part, section five, is devoted to the determination of minimal surfaces having an isothermal system composed of two families of curves each one of which has constant geodesic curvature. The only surfaces of this sort, besides B, are given by the Enneper-Weierstrass equations, where

$$F(u) = \frac{Ck}{(k - u^2)\sqrt{(k - u^2)(1 - ku^2)}}.$$

These surfaces are studied with the aid of elliptic functions.

The sixth and last section is given to the determination of minimal surfaces which are the envelopes of a family of similar cylinders whose elements are parallel to the xy plane and which may be brought into similar positions by rotation about the Z axis. Surfaces are found for which $F(u)$ has three different forms each depending on several parameters, and include the spiral minimal surfaces given by $F(u) = Cu^{-2+ni}$. where n is real. We note a slip at the bottom of page 174, where Stübler speaks of spiral minimal surfaces as given, in the notation of this article, by $F(u) = Cu^{ni}$, omitting -2 from the exponent.

II.

§ 4. Symmetries of B surfaces. There are certain properties of minimal surfaces applicable to surfaces of revolution with regard to symmetry which seem of sufficient interest and simplicity to deserve mention. Haag states in his first paper that he has studied the symmetries of these surfaces, but he gives no results.

Ribaucour states that the surface B_m consists of congruent nappes, and that it is brought into coincidence with itself by a rotation about the Z axis through the angle $2\pi/m$. It may be proved that if B_m is rotated about the Z axis through the angle $\pi + (\pi/m)$ it coincides with its symmetry with respect to the xy plane, so that if $m = (2a + 1)/(2b + 1)$, where a and b are integers, there are, as will appear later, other rotations besides those given by Ribaucour which bring the surface into coincidence with itself. The consideration of this and of other symmetries seems necessary for the proof that B_m and B_{-m} are identical and not only symmetrical surfaces, as also for the proof that all plane non-rectilinear geodesics (φ) are congruent and are p in number when $m = p/q$.

In order that B_m have symmetry with respect to the yz plane it is necessary and sufficient that $m = 2a/(2b + 1)$, where a and b are integers; symmetry with respect to the xy plane $m = (2a + 1)/(2b + 1)$; symmetry with respect to the origin $m = (2a + 1)/2b$; the surface is always symmetrical with respect to the xz plane since that plane contains a geodesic of the surface.

The sufficiency of these conditions is at once obvious on the substitution of $\pm u$, $\pm v$ for u, v in the equations of the surface. The necessity may be shown as follows: if, for example, B_m has symmetry with respect to the yz plane, the plane geodesic (φ), $u = v$, is symmetrical with respect to the Z axis. The equations of this curve are

$$x = \frac{r^{m-1}}{m-1} - \frac{r^{m+1}}{m+1}, \qquad y = 0, \qquad z = \frac{2r^m}{m}.$$

There must be a value r' giving $x' = -x$, $z' = z$. The last condition requires that $r = r'e^{2k\pi i/m}$, where k is an integer; the first that there be two integers, n and n', such that

$$2k\pi i \frac{m-1}{m} = (2n + 1)\pi i, \qquad 2k\pi i \frac{m+1}{m} = (2n' + 1)\pi i,$$

from which it follows that $m = 2a/(2b + 1)$. It may be proved that the straight lines (φ) lie in the planes of the plane geodesics (φ) if and only if $m = (2a + 1)/(2b + 1)$.

§ 5. Multiple points and lines. Results and discussion of the case $r_1 = r$. We shall prove in this and the following sections:

If $m > 1$, the plane of each plane geodesic (φ) of B_m contains a real double curve; all these double curves are congruent; the surface has no other real finite double curves; if $m = p/q$ there are p such curves; these curves are algebraic and of degree $\frac{1}{2}(p + q)(p + q - 1)$.

If $m < 1$, the plane of each plane geodesic (φ) contains a real double curve, and all these curves are congruent; if $m = p/q$ there are p such curves and their degree is $q(p + q - 1)$; if $m = p/q$ the straight lines φ, which lines are the intersection of B_m with the xy plane, are also double lines and are p in number.

If $m = 1$, the surface is periodic and contains an infinite number of congruent plane geodesics (φ) in equidistant planes parallel to the xz plane, and an infinite number of congruent plane double curves, one in each plane of a plane (φ). The xy plane is cut by B_1 in an infinite number of parallel straight lines (φ) bisecting the distances between these planes.

If $m = 0$, the surfaces are the minimal helicoids. The catenoid and the right helicoid have no real finite multiple points or lines. In all other cases the surface has an infinite number of double curves (r), which are circular helices, and along each of which two branches of the surface intersect at a constant angle.

In the determination of the real finite multiple points of the surfaces B_m we exclude at first the values $m = 1$ and $m = 0$. Writing, as before, $u = re^{\phi i}$ and $v = re^{-\phi i}$, we exclude $r = 0$, noting that this value gives

$$x = y = z = 0, \quad m > 1, \qquad x = y = \infty, \quad z = 0, \quad m < 1.$$

Real points of the surface are given in general by conjugate values of u, v or by real values of r and φ, and for such a point it is no restriction to suppose r positive. A *multiple point* is a point x, y, z given by more than one pair of values of u, v.

When x, y, z have fixed values, the same is true of the three expressions

$$x^2 + y^2 + \frac{m^2}{m^2 - 1} z^2 = \left(\frac{r^{m-1}}{m-1} + \frac{r^{m+1}}{m+1} \right)^2, \qquad x + iy = -\frac{u^{m+1}}{m+1} + \frac{v^{m-1}}{m-1},$$

$$\frac{mz}{2} = r^m \cos m\varphi.$$

Conversely, if for more than one pair of values of u, v or r, φ these expressions have the same values, the coördinates x, y, z have the same values, if the latter are real and finite.

We enquire first if a point may be given by two pairs of coördinates, r, φ and r_1, φ_1, where $r_1 = r \mp 0$, $\varphi_1 \mp \varphi + 2n\pi$. From the value of z it follows $\cos m\varphi_1 = \cos m\varphi$, $\varphi_1 = (2k\pi/m) \pm \varphi$.

From $x + iy$, choosing $\varphi_1 = (2k\pi/m) - \varphi$,

$$-\frac{r^2}{m+1} e^{(m+1)[(2k\pi i/m) - \phi i]} + \frac{1}{m-1} e^{-(m-1)[(2k\pi i/m) - \phi i]}$$
$$= -\frac{r^2}{m+1} e^{(m+1)\phi i} + \frac{1}{m-1} e^{-(m-1)\phi i}.$$

This may be reduced to

$$\frac{r^2}{m+1} \sin \left[(m+1)\left(\varphi - \frac{k\pi}{m} \right) \right] + \frac{1}{m-1} \sin \left[(m-1)\left(\varphi - \frac{k\pi}{m} \right) \right] = 0.$$

This becomes, for $k = 0$,

$$\frac{r^2}{m+1} \sin (m+1)\varphi + \frac{1}{m-1} \sin (m-1)\varphi = 0.$$

This is exactly the equation, $y = 0$, from which $-r^{m-1}$ has been cancelled. Every point of B_m in this plane is given by two pairs of coördinates, r, φ and r, $-\varphi$, or by u, v and v, u; the xz plane cuts the surface in the plane geodesic line of curvature, $u = v$, which is not a double curve, the two pairs of coördinates being in this case identical, and in a double curve for which $u \neq v$. From the values of X, Y, Z it is clear that the normals to the two branches of the surface at a point of this curve make equal angles with the X and Z axes and supplementary angles with the Y axis. If $m = p/q$, the equation of the double curve is found by cancelling from the equation above $\sin(\varphi/q)$. When $m = p/q$, the degree of the double curve is one half the difference of the degree of the surface and that of the plane geodesic, $u = v$. The degree is

$$\tfrac{1}{2}[(p+q)^2 - (p+q)] = \tfrac{1}{2}(p+q)(p+q-1), \qquad m > 1,$$

$$\tfrac{1}{2}[2q(p+q) - 2q] = q(p+q-1), \qquad m < 1.$$

When $m = (2a+1)/(2b+1)$ the straight line (φ), in this case the X axis, is part of the double curve.

Since B_m is brought into coincidence with itself or a symmetrical surface by a rotation about the Z axis through angles π/m there is a congruent double curve in the plane of each plane geodesic (φ); all these double curves are given by $\varphi_1 = (2k\pi/m) - \varphi$. When $m = p/q$ the number of these double lines is p.

We now form a similar equation from $x + iy$, taking $\varphi_1 = (2k\pi/m) + \varphi$. From this equation the factor $1 - e^{2k\pi i/m}$ may be cancelled. It becomes

$$\frac{r^2}{m+1} e^{(m+1)\phi i} - \frac{1}{m-1} e^{-(m-1)\phi i} = 0.$$

This is the equation $x + iy = 0$; we rewrite it

$$r^2 = \frac{m+1}{m-1}(\cos 2m\varphi - i \sin 2m\varphi).$$

If r is real, $2m\varphi = n\pi$, where n is an integer even or odd as m is greater or less than 1.

Suppose $m > 1$, then

$$r = \sqrt{\frac{m+1}{m-1}}, \qquad \varphi = \frac{n'\pi}{m}, \qquad n = 2n', \qquad x = y = 0,$$

$$z = \pm \frac{2}{m}\left(\frac{m+1}{m-1}\right)^{m/2},$$

the upper or lower sign being chosen in z as n' is even or odd. These are the points in which the plane geodesics (φ) cut the Z axis. The point or points in which the geodesic (φ) lying in the xz plane cuts the Z axis are given by

$$r = \sqrt{\frac{m+1}{m-1}}, \qquad \varphi = h\pi, \qquad h \text{ integral.}$$

We conclude that this geodesic passes through both points on the Z axis if $m = p/q$ and p is odd, in other cases through one of the points. The double curve in the xz plane goes through both points on the Z axis except for $m = 2$, Enneper's surface, in which case it goes through the point not on the geodesic (φ) in that plane.

Suppose $m < 1$, then

$$r = \sqrt{\frac{1+m}{1-m}}, \qquad \varphi = \frac{2n'+1}{m}\frac{\pi}{2}, \qquad n = 2n'+1, \qquad x = y = z = 0.$$

The geodesic (φ) of the xz plane does not contain the origin, hence every double line of those considered does pass through the origin. It is of some interest to consider

$$r = \sqrt{\frac{1+m}{1-m}}, \qquad \varphi = \frac{2n'+1}{m}\frac{\pi}{2}, \qquad m > 1,$$

$$r = \sqrt{\frac{m+1}{m-1}}, \qquad \varphi = \frac{n'\pi}{m}, \qquad m < 1.$$

These coördinates give to u and v values which are not conjugate, so that X, Y, Z are not all real. If such coördinates give real values to x, y, z that point must be considered an isolated singular point of a real curve whose equation is satisfied by such coördinates r, φ. The first pair gives the origin, which is accordingly an isolated singular point of each double curve. The second pair gives again the points in which the plane geodesics (φ) cut the Z axis, but they are real only if $m = 2a/(2b+1)$. In this case one lies on each plane geodesic (φ). Both lie on each double line unless $p = 2$; if $p = 2$ then, as in the case of Enneper's surface, there are but two plane geodesics (φ) and two double curves; each has as an isolated singular point that one of the two points which is not an isolated singular point of the plane geodesic (φ) in the same plane.

We consider again the case $m = (2a+1)/(2b+1)$, where the straight lines (φ) form part of the double curves. We have here, as in other particulars, a striking difference in the cases $m > 1$ and $m < 1$. The line in the xz plane, that is the X axis, is given by $\varphi = \pm (\pi/2)$; we find

$$y = z = 0, \qquad x = \left(\frac{r^{m-1}}{m-1} + \frac{r^{m+1}}{m+1}\right)\cos(a+b)\pi.$$

If $m > 1$, one half-line issuing from the origin is given by positive real values of r, the other half-line only by complex values of r. One half-line is therefore the intersection of two real branches of the surface while the other half-line is not. If $m < 1$, all points of the X axis are given by positive real values of r and two real branches of the surface intersect all along the line.

§ 6. **Double lines $r_1 \neq r$.** We now consider the possibility of multiple points for which $r_1 \neq r$. We must have

$$\left(\frac{r_1^{m-1}}{m-1}+\frac{r_1^{m+1}}{m+1}\right)^2=\left(\frac{r^{m-1}}{m-1}+\frac{r^{m+1}}{m+1}\right)^2.$$

If $m > 1$, this equation can be satisfied only by $r_1 = r$ or $r_1 = -r$. The first case has been considered; the second can be reduced to it, for if $u_1 = r_1 e^{\phi_1 i}$ and $v_1 = r_1 e^{-\phi_1 i}$, $r_1 = -r$, we may write $u_1 = re^{(\phi_1+\pi)i}$ and $v_1 = re^{-(\phi_1+\pi)i}$. In this case B_m has no other real finite multiple points or lines than those already given. If $m < 1$, the equation above for r_1 is always satisfied by one and only one value $r_1 \neq \pm r$. Supposing r_1 to have this value, we have, for rational values of m, for

$$m=\frac{2a}{2b+1}, \qquad \varphi=\frac{2n+1}{m}\frac{\pi}{2}, \qquad \varphi_1=\varphi+(2b+1)\pi,$$

$$\frac{r_1^{m-1}}{m-1}+\frac{r_1^{m+1}}{m+1}=-\left(\frac{r^{m-1}}{m-1}+\frac{r^{m+1}}{m+1}\right),$$

$z = 0$ and the same values for x and y, giving points on the straight lines (φ). Two real branches of the surface intersect along each of these lines. The normals to the two branches at a point of such a line are not generally equally inclined to any of the coördinate axes. When

$$m=\frac{2a+1}{2b}, \qquad \varphi=\frac{2n+1}{m}\frac{\pi}{2}, \qquad \varphi_1=\varphi+2b\pi,$$

$$\frac{r_1^{m-1}}{m-1}+\frac{r_1^{m+1}}{m+1}=-\left(\frac{r^{m-1}}{m-1}+\frac{r^{m+1}}{m+1}\right),$$

give the same result. The straight lines (φ) are in this case also double lines all along which two real branches intersect, the normals to the two branches making generally different angles with all the axes.

When $m = (2a + 1)/(2b + 1)$, if $r_1 \neq \pm r$, there is no value of φ_1 such that the points on the straight lines (φ) are given by the two pairs of coördinates r, φ and r_1, φ_1, but these lines have already been proved to be double lines, each point of such a line being given by two pairs of coördinates, r, φ and r, $-\varphi$. In this case we have each point x, y, 0 on such a line given by r, $\varphi = (2n + 1)/m\,(\pi/2)$, and the point $-x$, $-y$, 0, symmetrical with respect to the origin, given by $r_1 \neq \pm r$, $\varphi_1 = \varphi + (2b + 1)\pi$.

Ribaucour showed that each curve (r) lies on a quadric of revolution. If $m < 1$, there are on each of these quadrics, hyperboloids of one sheet in this case, two curves, (r) and (r_1), $r_1 \neq \pm r$; if $m = 2a/(2b + 1)$ or $(2a + 1)/2b$ these curves intersect on the straight lines (φ); if $m = (2a + 1)/(2b + 1)$, the curves (r) and (r_1) meet the xy plane at points on the straight lines (φ) symmetrical with respect to the origin.

§ 7. **The surfaces B_1.** Writing $u = re^{\phi i}$ and $v = re^{-\phi i}$ the equations of B_1 are

$$x = \log r - \tfrac{1}{2}r^2 \cos 2\varphi, \qquad y = -\varphi - \tfrac{1}{2}r^2 \sin 2\varphi,$$
$$z = 2r \cos \varphi.$$

The surface is obviously periodic, one period being given by

$$2\pi > \varphi \geqq 0, \qquad 2\pi > -y \geqq 0.$$

The value $m = -1$ gives the same surface in a different position. The surface has an infinite number of congruent plane geodesic lines of curvature (φ) in equidistant planes, parallel to the xz plane; the equations are $u = v$, or $\varphi = -n\pi$, or

$$x = \log r - \tfrac{1}{2}r^2, \qquad y = n\pi, \qquad z = \pm 2r.$$

The xy plane cuts the surface in an infinite number of parallel straight lines (φ) whose equations are $u = -v$, or $\varphi = -(2n + 1)(\pi/2)$, or

$$x = \log r \pm \tfrac{1}{2}r^2, \qquad y = (2n + 1)\frac{\pi}{2}, \qquad z = 0.$$

The curves (r) lie on the parabolic cylinders

$$z^2 + 4x = 2(\log r^2 + r^2).$$

As for B_m in general the curves (φ) are helices on similar cylinders whose elements are parallel to the xy plane.

We seek the real finite double lines. Any point x, y, z is given by only one value of $r \neq 0$ since $z^2 + 4x = 2(\log r^2 + r^2)$. If φ_1 and φ give the same point, we have from the value of z

$$\cos \varphi_1 = \cos \varphi, \qquad \varphi_1 = -2n\pi \pm \varphi.$$

The value of y gives

$$2n\pi \mp \varphi \mp \tfrac{1}{2}r^2 \sin 2\varphi = -\varphi - \tfrac{1}{2}r^2 \sin 2\varphi.$$

The upper signs give $2n\pi = 0$, $\varphi_1 = \varphi$; the lower signs

$$y = -\varphi - \tfrac{1}{2}r^2 \sin 2\varphi = n\pi.$$

There are an infinite number of congruent double curves lying in the planes $y = n\pi$. Choosing $n = 0$ the two pairs of coördinates of a point of this curve are r, φ and r, $-\varphi$, $\varphi \mp k\pi$. The normals to the two branches intersecting in this line make equal angles with the X and Z axes, supplementary angles with the Y axis.

§ 8. **The surfaces B_0.** We may without restriction write for the minimal helicoids, $m = 0$,

$$F(u) = -\frac{e^{\alpha i}}{u^2}, \qquad \phi(v) = -\frac{e^{-\alpha i}}{v^2}.$$

The equations of the surface are

$$x = r\cos(\varphi + \alpha) + \frac{1}{r}\cos(\varphi - \alpha), \qquad y = r\sin(\varphi + \alpha) + \frac{1}{r}\sin(\varphi + \alpha),$$

$$z = 2\varphi\sin\alpha - 2\log r\cos\alpha.$$

The value $\alpha = 0$ gives the catenoid; $\alpha = \pi/2$ the right helicoid. For all other surfaces we may suppose $\pi/2 > \alpha > 0$, since the surfaces corresponding to $\pi \pm \alpha$ are symmetrical to the surface α. The equations may also be conveniently written so as to involve only trigonometric functions by putting $r = \operatorname{ctn}(\rho/2)$,* or so as to involve trigonometric and hyperbolic functions by putting $\log r = \rho$.† Among the geodesics (φ) are no plane curves and no lines of curvature except for the catenoid, where they are the meridians, and no asymptotic lines except for the right helicoid, where they are the rulings. The curves (r) are lines of curvature and asymptotic lines in the same cases, and are plane only for the catenoid.

The curves (r) are circular helices lying on the cylinders

$$x^2 + y^2 = r^2 + \frac{1}{r^2} + 2\cos 2\alpha, \qquad \cos\gamma = \frac{2r\sin\alpha}{r^2 + 1},$$

where γ is the angle which the tangent makes with the Z axis. The helix $r = 1$ is on the cylinder of minimum radius

$$x^2 + y^2 = 4\cos^2\alpha$$

and has the maximum pitch, $\gamma = (\pi/2) - \alpha$. It is Stübler's curve of maximum total curvature, and the minimal helicoid is tangent to the circular cylinder along it, so that it is a geodesic of the helicoid. On each of these cylinders whose radius is greater than $2\cos\alpha$ there are, with certain exceptions to be given presently, two helices, (r) and $(1/r)$, which have the same pitch. The geodesics (φ) lie on equal hyperbolic cylinders whose elements are parallel to the Z axis, and whose envelope is the cylinder $x^2 + y^2 = 4\cos^2\alpha$. The eccentricity of the hyperbola directrices is sec α. As for B_m in general the geodesics (φ) are the curves of contact of the minimal helicoid and similar cylinders whose elements are parallel to the xy plane.

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* See Stübler's paper, p. 166.

† Bianchi, loc. cit., p. 324.

We seek the multiple lines of the surface. It is obvious, and it may also be easily shown from the equations, that the catenoid, $\alpha = 0$, and the right helicoid, $\alpha = \pi/2$, have no real finite double curve or point. The situation is different when α is between 0 and $\pi/2$.

Since $x^2 + y^2 = r^2 + (1/r^2) + 2 \cos 2\alpha$ a point x, y, z can be given only by r and $r_1 = 1/r$. If r is fixed it follows from the value of z that the same is true of φ. If $r_1 = 1/r$ and φ_1 give the same point as r, φ we have

$$x = \frac{1}{r} \cos (\varphi_1 + \alpha) + r \cos (\varphi_1 - \alpha) = r \cos (\varphi + \alpha) + \frac{1}{r} \cos (\varphi - \alpha),$$

$$y = \frac{1}{r} \sin (\varphi_1 + \alpha) + r \sin (\varphi_1 - \alpha) = r \sin (\varphi + \alpha) + \frac{1}{r} \sin (\varphi - a),$$

$$z = 2\varphi_1 \sin \alpha + 2 \log r \cdot \cos \alpha = 2\varphi \sin \alpha - 2 \log r \cdot \cos \alpha.$$

The last equation gives

$$\varphi_1 - \varphi = - 2 \log r \cdot \operatorname{ctn} \alpha.$$

The first two are equivalent to the single equation

$$r^2 = \frac{\sin\left(\alpha + \frac{\varphi_1 - \varphi}{2}\right)}{\sin\left(\alpha - \frac{\varphi_1 - \varphi}{2}\right)}.$$

Then r must satisfy the equation

$$r^2 = \frac{\sin (\alpha - \log r \cdot \operatorname{ctn} \alpha)}{\sin (\alpha + \log r \cdot \operatorname{ctn} \alpha)}.$$

Conversely, if r is any solution of this equation the coördinates r, φ and $r_1 = 1/r$, $\varphi_1 = \varphi - 2 \log r \cdot \operatorname{ctn} \alpha$ give the same values for x, y, z.

We consider the solutions of the equation for r. It is clear that $r = 1$ is a solution. This value, as already noted, gives the cylinder of minimum radius, which is tangent to the helicoid along this helix. The latter is not a double curve, for evidently $r_1 = r$ and $\varphi_1 = \varphi$. Further it is clear that if r is a solution of the equation $1/r$ is also a solution. If we write $\lambda = \log r \operatorname{ctn} \alpha$, the equation can be written

$$\tan \lambda + \tan \alpha \tanh (\lambda \tan \alpha) = 0.$$

It may be proved that this equation for λ has an infinite number of real roots, one and only one λ_n for each integral value of n, and such that

$$n\pi > \lambda_n > n\pi - \alpha.$$

As n is increased $\lambda_n - (n\pi - \alpha)$ decreases and approaches zero with great rapidity, at least as rapidly as $e^{-2\pi n}$. The values $r = e^{\mp\lambda_n \tan\alpha}$ give the same helix, a double curve, and the circular cylinders on which these lie carry each only one helix. For every curve (r) the tangent plane along the curve makes a constant angle with the xy plane, these curves being the Minding parallels of the helicoid. The same thing is of course true for each of the branches of the surface intersecting in a double curve; it appears that at each point the normals to the two branches make supplementary angles with the Z axis, and moreover form a constant angle all along the curve.

§ 9. **A new property characteristic of *B* surfaces.** We state without proof a new property characteristic of minimal surfaces applicable to surfaces of revolution. It is planned to publish a proof of this property on another occasion.

Schwarz proved* that corresponding points, that is points corresponding to the same values of the parameters u, v, of a family of associate minimal surfaces lie on an ellipse. The characteristic property is this:

The necessary and sufficient condition that the envelope of a family of minimal surfaces associate to a given minimal surface be a part of the locus of the extremities of the principal axes of Schwarz's ellipses, with coincidence of corresponding points, is that the given minimal surface be applicable to a surface of revolution, that is a surface B_m, where m is not 1 or 0.

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December 13, 1916.

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* Loc. cit., p. 295.